

1. Are primes plentiful or scarce?

What do you think?

We will consider both points of view:

1. The primes are scarce.
2. The primes are plentiful.

3. Prime Deserts

A group of n consecutive composite integers is called a **prime desert** of length n .

Show that

$$24, 25, 26, 27, 28$$

is a prime desert of length 5.

Show that

$$90, 91, 92, 93, 94, 95, 96$$

is a prime desert of length 7.

How long can prime deserts be?

5. Shoot for 100

Problem Find 100 consecutive composite numbers.

Answer: Let

$$x = 1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times 100 \times 101.$$

Then the numbers

$$x + 2, x + 3, x + 4, \dots, x + 101$$

are all composite.

7. Arbitrarily Long Prime Deserts

For any integer n , prove there exists n consecutive composite numbers.

Consequently, there exist **arbitrarily long** prime deserts.

What does “arbitrarily long” mean?

Conclusion: Primes are scarce.

9. Proving Euclid’s Theorem

Euclid’s Theorem To see why Euclid’s Theorem is true, use Katie’s Theorem with

$$x = n! = 1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times n.$$

Explain how Katie’s Theorem shows that no prime less than n divides $x + 1$.

2. Another Solution

Another solution to Charise’s problem is the number

$$x = 5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$$

This would be a very old Grandpaw.

4. Connection to Charise’s Problem

Charise’s Problem Since the numbers

$$120 + 2, \quad 120 + 3, \quad 120 + 4, \quad 120 + 5$$

are divisible, respectively, by 2, 3, 4, and 5,

the number $122 = 5! + 2$ is the start of a prime desert of length 4.

6. How do you know?

To see this, simply observe that

2 divides $x + 2$

3 divides $x + 3$

4 divides $x + 4$

$\vdots \quad \vdots \quad \vdots$

101 divides $x + 101$

What is the length of this list?

8. Euclid’s Theorem: the primes abide

Euclid’s Theorem Over two thousand years ago Euclid (the geomtry dude) showed that we never run out of primes.

Euclid’s Theorem. Given any positive integer n , there is always a prime p larger than n .

10. Finding new primes

For $n = 2, 3, 4, 5$, compute $n! + 1$.

What new primes divide $n! + 1$?

11. An Infinity of Primes

Because, no matter how large n is, there is a prime number bigger than n , you never run out of primes, which proves

Theorem. There are infinitely many primes.

Since there are **infinitely** many primes, they must be plentiful.

13. Enter Probability

The question “Are the primes plentiful or scarce?” really boils down to:

If I pick a large integer (at random) what is the probability it will be prime?

This question has an answer.

15. Application to Encryption

Encryption RSA RSA and many security systems and secret codes require large primes around 100 digits long.

In order to implement these codes, we need to know how likely it is that a 100 digit number chosen at random will be prime.

By the Prime Number Theorem, the answer is roughly $1/230$.

If you avoid even numbers, the probability increases to $1/115$. Why?

17. Twin Primes

Primes sometimes come in pairs, two apart:

3 and 5

5 and 7

11 and 13

17 and 19

$33218925 \times 2^{169690} - 1$ and $33218925 \times 2^{169690} + 1$

This last example has more than 51,000 digits!

Such numbers are called **twin primes**.

It is unknown (that is, unproven) whether there are infinitely many twin primes.

12. Second thought

Wait a minute, there are infinitely many powers of 10 (10, 100, 1000, 10000, etc.) but these do not seem very plentiful.

There are only 12 powers of 10 up to one trillion.

We want to know: Are big primes as rare as finding a parking space on campus?

or as winning the lottery?

14. The Prime Number Theorem

Prime Number Theorem Pick a number at random between 1 and some large bound N .

Then the probability the number you chose is prime is approximately 1 out of 2.3 times the number of digits of N .

16. Primes of form $4x + 1$

Primes of form $4x + 1$

Examine the table $y = 4x + 1$, searching for primes.

x	$4x + 1$	Prime or Composite?
1	5	Prime
2	9	Composite
3	13	Prime
4	17	Prime
5	21	Composite
6	25	Composite
7	29	Prime
8	33	Composite
9	37	Prime
10	41	Prime

It has been proven that this list consists of infinitely many primes.

18. Primes of form $x^2 + 1$

Primes of form $x^2 + 1$ If you examine the table for $y = x^2 + 1$, you will find many primes.

You need only consider when x is even. Why?

x	$x^2 + 1$	Prime or Composite?
2	5	Prime
4	17	Prime
6	37	Prime
8	65	Composite
10	101	Prime

No one knows whether there are infinitely many primes in this list.