

THE RACE TO INFINITY

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1. BEHAVIOR OF x^n , e^x AND $\ln x$

If a and b are positive constants, then

$$(1) \lim_{x \rightarrow \infty} \frac{(\ln x)^b}{x^a} = 0$$

and

$$(2) \lim_{x \rightarrow \infty} \frac{x^b}{e^{ax}} = 0.$$

That is, in a race to infinity between these three functions, the outcome will be:

First place: e^{ax}

Second place: x^b

Third place: $(\ln x)^c$

no matter what values we make the constants a , b , and c .

2. PROOF

A simple proof may be given directly from the integral definition of the natural logarithm.

Let c be any positive constant.

(The value of c will be determined later.)

Then for all $t \geq 1$ we have

$$\frac{1}{t} = t^{-1} \leq t^{c-1}.$$

It follows that for $x > 1$

$$\int_1^x \frac{1}{t} dt \leq \int_1^x t^{c-1} dt.$$

Performing the integration we find

$$0 < \ln x \leq \frac{x^c - 1}{c} < \frac{x^c}{c}.$$

3. PROOF - PART 2

$$0 < \ln x < \frac{x^c}{c}.$$

Raising both sides to the power b we get

$$0 < (\ln x)^b < \frac{x^{bc}}{c^b}.$$

Now divide by x^a to get our final inequality:

$$0 < \frac{(\ln x)^b}{x^a} < \frac{x^{bc-a}}{c^b}.$$

This holds for any positive constant c .

4. PROOF - FINISH

$$0 < \frac{(\ln x)^b}{x^a} < \frac{x^{bc-a}}{c^b}.$$

It is now time to select an appropriate value for c .

$$\text{Choose } c = \frac{a}{2b}.$$

Then x^{bc-a} becomes $x^{-a/2} = \frac{1}{x^{a/2}}$,

which approaches 0 as x approaches infinity.

By the Squeeze Theorem, it follows that

$\frac{(\ln x)^b}{x^a}$ must also tend to 0 as $x \rightarrow \infty$.

This proves statement (1).

5. STATEMENT (2)

Recall

$$(1) \lim_{x \rightarrow \infty} \frac{(\ln x)^b}{x^a} = 0$$

$$(2) \lim_{x \rightarrow \infty} \frac{x^b}{e^{ax}} = 0.$$

Statement (2) follows easily from (1).

Just make the change of variable $t = e^x$.

Then $x = \ln t$ and so

$$\frac{x^b}{e^{ax}} = \frac{(\ln t)^b}{(e^{\ln t})^a} = \frac{(\ln t)^b}{t^a},$$

which approaches 0 as x (and hence $t = e^x$) goes to infinity.

This proves statement (2).