

ACTIVITY 8. THE RUTH–AARON NUMBERS

At the end of his baseball career in 1935 Babe Ruth hit a total of 714 home runs, a record that would last until 1974 when Hank Aaron hit homerun number 715. The numbers 714 and 715 have some interesting properties when we factor them into primes. (For one thing we know, from Katie’s Theorem, that the prime divisors of 714 and 715 will not overlap since 714 and 715 are consecutive integers.) A little division shows

$$714 = 2 \times 3 \times 7 \times 17$$

$$715 = 5 \times 11 \times 13$$

(1) If we combine the prime factors of both numbers, we get all the primes up to 17: 2, 3, 5, 7, 11, 13, 17.

(2) If we add the prime divisors of 714 and 715 we get the same sum:

$$2 + 3 + 7 + 17 = 29$$

$$5 + 11 + 13 = 29.$$

Phenomenon (1). Find three examples up to $n = 25$, where the numbers n and $n + 1$ contain all the primes up to the largest prime divisor of n and $n + 1$.

Phenomenon (2). Let $S(n)$ = the sum of the prime divisors of n . We do not count repetition of primes, for example, $S(12) = 2 + 3 = 5$, even though $12 = 2 \times 2 \times 3$. Since $n = 1$ has no prime divisors, we take $S(1) = 0$.

Two consecutive numbers $(n, n + 1)$ are **Ruth–Aaron numbers** if and only if $S(n) = S(n + 1)$.

Write the value of $S(n)$ in Tables 1 and 2 below. (The first four entries are already filled in for you.)

Table 1. $S(n)$ for $n = 1$ to 50.

n	$S(n)$	n	$S(n)$	n	$S(n)$	n	$S(n)$	n	$S(n)$
1	0	11	—	21	—	31	—	41	—
2	2	12	—	22	—	32	—	42	—
3	3	13	—	23	—	33	—	43	—
4	2	14	—	24	—	34	—	44	—
5	—	15	—	25	—	35	—	45	—
6	—	16	—	26	—	36	—	46	—
7	—	17	—	27	—	37	—	47	—
8	—	18	—	28	—	38	—	48	—
9	—	19	—	29	—	39	—	49	—
10	—	20	—	30	—	40	—	50	—

Table 2. $S(n)$ for $n = 51$ to 100.

n	$S(n)$	n	$S(n)$	n	$S(n)$	n	$S(n)$	n	$S(n)$
51	—	61	—	71	—	81	—	91	—
52	—	62	—	72	—	82	—	92	—
53	—	63	—	73	—	83	—	93	—
54	—	64	—	74	—	84	—	94	—
55	—	65	—	75	—	85	—	95	—
56	—	66	—	76	—	86	—	96	—
57	—	67	—	77	—	87	—	97	—
58	—	68	—	78	—	88	—	98	—
59	—	69	—	79	—	89	—	99	—
60	—	70	—	80	—	90	—	100	—

Examine Tables 1 and 2 looking for consecutive integers n and $n + 1$, where $S(n) = S(n + 1)$. These are the Ruth–Aaron Numbers up to 100.

The next two Ruth–Aaron Numbers after (714,715) are: (1682, 1683) and (2107, 2108). It seems unlikely that any one will hit this many home runs.

It is known that there are infinitely many pairs of Ruth–Aaron numbers.