

Surface Area of a Sphere

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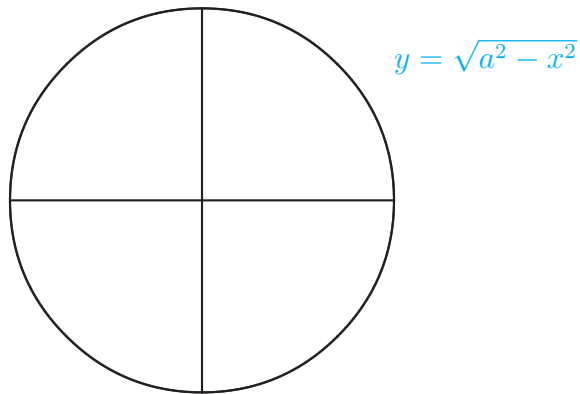
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1. SPHERE OF RADIUS a

Equation: $x^2 + y^2 = a^2$

x ranges from 0 to a .

$$y = \pm\sqrt{a^2 - x^2}$$



2. FIND THE DERIVATIVE $\frac{dy}{dx}$

Differentiate both sides of

$$x^2 + y^2 = a^2$$

w.r.t. x :

$$2x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

3. SURFACE AREA FORMULA

Double the surface area of a hemisphere of radius a .

$$\begin{aligned} A &= 2 \cdot 2\pi \int_0^a y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= 4\pi \int_0^a y \sqrt{1 + \left(\frac{-x}{y}\right)^2} dx \\ &= 4\pi \int_0^a y \sqrt{1 + \frac{x^2}{y^2}} dx = 4\pi \int_0^a y \sqrt{\frac{x^2 + y^2}{y^2}} dx \\ &= 4\pi \int_0^a y \sqrt{\frac{a^2}{y^2}} dx = 4\pi \int_0^a y \frac{a}{y} dx \\ &= 4\pi \int_0^a a dx = 4\pi ax \Big|_0^a = 4\pi a^2 \end{aligned}$$

4. FIND CIRCUMFERENCE OF A CIRCLE OF RADIUS a

Find the arclength of the quarter circle in the first quadrant and multiply by 4.

The integral is the same as the last integral for surface area of a sphere, but without the term y .

$$L = 4 \int_0^a \sqrt{1 + \left(\frac{-x}{y}\right)^2} dx$$

$$\begin{aligned}
&= 4 \int_0^a \sqrt{1 + \frac{x^2}{y^2}} dx \\
&= 4 \int_0^a \sqrt{\frac{x^2 + y^2}{y^2}} dx \\
&= 4 \int_0^a \frac{a}{y} dx = 4a \int_0^a \frac{dx}{\sqrt{a^2 - x^2}}
\end{aligned}$$

5. EVALUATE THE INDEFINITE INTEGRAL

$$\text{Circumference} = 4a \int_0^a \frac{dx}{\sqrt{a^2 - x^2}}$$

Now what?

We want to find the general integral

$$I = \int \frac{dx}{\sqrt{a^2 - x^2}}$$

It turns out that trig saves the day.

Magic substitution: $x = a \sin t$

$$\frac{dx}{dt} = a \cos t$$

$$dx = a \cos t \, dt$$

6. FINISHING THE INTEGRATION

$$\begin{aligned}
I &= \int \frac{dx}{\sqrt{a^2 - x^2}} \\
&= \int \frac{a \cos t \, dt}{\sqrt{a^2 - (a \sin^2 t)}} \\
&= \int \frac{a \cos t \, dt}{\sqrt{a^2(1 - \sin^2 t)}}
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{a \cos t \, dt}{a \sqrt{1 - \sin^2 t}} \\
&= \int \frac{a \cos t \, dt}{a \sqrt{\cos^2 t}} \\
&= \int \frac{a \cos t \, dt}{a \cos t} = \int dt = t + C
\end{aligned}$$

Can this be true?

7. FINISHING THE PROBLEM

In our substitution: $x = a \sin t$,

as x goes from 0 to a ,

t goes from 0 to $\pi/2$

$$\begin{aligned}
\text{So } C &= 4a \int \frac{dx}{\sqrt{a^2 - x^2}} \\
&= 4a \int_0^{\pi/2} dt = 4a t \Big|_0^{\pi/2} \\
&= 4a \frac{\pi}{2} = 2\pi a
\end{aligned}$$

8. DISCUSSION QUESTIONS

1. Very rarely does the integral in the arclength or surface area problem work out nicely.
2. How could we find arclength when the integral is nasty?
3. Even for circumference of a circle, the integral was surprisingly hard.
4. There is a question of circularity. We knew what arclength was in order to define radian measure.
5. How could we remove this problem of circularity?