

THINGS TO KNOW FOR THE FINAL

- (1) Incidence axioms and distance axioms
- (2) Definition of Betweenness $A - B - C$
- (3) Unique Middle Theorem
- (4) Triangle Inequality on a line
- (5) Betweenness of Points Axiom
- (6) Quadrichotomy Axiom
- (7) Segment $\overline{AB}$ and ray $\overrightarrow{AB}$
- (8) Endpoint of a segment or ray
- (9) Ruler (or Real Ray) Axiom
 Given a ray $\overrightarrow{AB}$ and a real number $0 \leq r \leq \omega$. There is a unique point X on $\overrightarrow{AB}$ so that $AX = r$.
- (10) Midpoint Theorem. Every segment $\overline{AB}$ has a unique midpoint.
- (11) “Two points on a ray” Theorem
 Given two distinct points X and Y other than A on ray $\overrightarrow{AB}$. Then either $A - X - Y$ or $A - Y - X$.
- (12) Definition of Opposite Rays
- (13) Opposite Rays Theorem
 If $B - A - C$, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays. Moreover, for $\ell = \overleftrightarrow{AB}$,

$$\overrightarrow{AB} \cap \overrightarrow{AC} = \begin{cases} \{A\} & \text{when } \omega = \infty \\ \{A, A_\ell^*\} & \text{when } \omega < \infty \end{cases}$$
- (14) Separation Axiom Any line ℓ splits $\mathbb{P}$ into 3 disjoint sets H , K , and ℓ itself. The sets H and K are called (opposite) halfplanes and they have the properties:
 - (i) If X and Y both lie in H (or K), then $\overline{XY}$ lies entirely in H (or K);
 - (ii) If X lies in H and Y lies in K , then segment $\overline{XY}$ meets ℓ .
 On the sphere, the two sides are called hemispheres and Y is not allowed to be X^* in properties (i) and (ii).
- (15) Same side and opposite side of a line
- (16) Antipode on a line A_ℓ^* Theorem
- (17) Antipode of a point A^* Theorem
- (18) Definition of betweenness for rays
- (19) Angle measurement axioms for the angle between coterminal rays
 - (a) $m\angle ABC$ is a real number between 0 and 180
 - (b) $m\angle ABC = 0$ exactly when $\angle ABC$ is the zero angle
 - (c) $m\angle ABC = 0$ exactly when $\angle ABC$ is a straight angle
 - (d) symmetry: $m\angle ABC = m\angle CBA$

- (20) Protractor (or Real Fan) Axiom
Given a ray $\overrightarrow{BA}$ and a halfplane H for $\overleftrightarrow{AB}$ and a real number s in the range $0 \leq s \leq 180$. There is a unique ray $\overrightarrow{BX}$ lying in H so that $m\angle XBA = s$.
- (21) Duality
- (22) Angle bisector Theorem. Every proper angle has a unique bisector.
- (23) Special angles:
 - (a) zero angle
 - (b) right angle
 - (c) acute angle
 - (d) obtuse angle
 - (e) straight angle
- (24) supplementary angles, complementary angles
- (25) interior of an angle
- (26) definition of triangle $\triangle ABC$
- (27) interior of a triangle
- (28) congruent segments, congruent angles, congruent triangles
- (29) SAS Axiom
If under the correspondence $ABC \leftrightarrow XYZ$ between the vertices of the triangles $\triangle ABC$ and $\triangle XYZ$, two sides and the included angle of the first triangles are congruent, respectively, to two sides and the included angle of the second triangle, then $\triangle ABC \cong \triangle XYZ$.
- (30) Special triangles:
 - (a) isosceles triangle
 - (b) equilateral triangle
 - (c) scalene triangle
 - (d) right triangle
 - (e) legs of a right triangle
 - (f) hypotenuse of a right triangle
- (31) Angle–Side–Angle
- (32) Angle–Angle–Side
- (33) Side–Side–Side
- (34) Compatability Axiom
- (35) The Crossbar Theorem
- (36) Special lines:
 - (a) perpendicular lines
 - (b) parallel lines
 - (c) transversal
- (37) Existence of Perpendicular Lines

Raising a Perpendicular. Given a line ℓ and a point P on ℓ , there is a unique line m through P and perpendicular to ℓ .

Dropping a Perpendicular. Given a line ℓ and a point P not on ℓ , you can always find a line m through P which is perpendicular to ℓ .

When $\omega = \infty$ ($\mathbb{E}$ and $\mathbb{H}$), line m is always unique.

For $\mathbb{S}$, line m is unique *unless* the distance from P to any point on ℓ is $\omega/2 = \pi r/2$, in which case all lines through P are perpendicular to ℓ .

[Think of P as the North pole and ℓ as the equator.]

(38) Parallel Postulate and Euclid's 5th Postulate

(39) transversal configuration, alternate interior angles

(40) angle sum

EUCLIDEAN VERSUS NON-EUCLIDEAN GEOMETRY:

Euclidean plane $\mathbb{E}$:

Distance formula: If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then

$$e(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Moreover, $e(P, Q) = \sqrt{1 + m^2} |x_1 - x_2|$ if P and Q lie on line $y = mx + b$.

Betweenness Axioms: Given three distinct collinear points A, B, C , exactly one of them is between the other two.

angle measure

opposite halfplane

Parallel Postulate holds in $\mathbb{E}$.

In $\triangle ABC$, $m\angle A + m\angle B + m\angle C = 180$.

The sphere $\mathbb{S}$:

$\omega = \pi r$, where r is the radius of the sphere

lines are great circles

every point A has an antipode A^*

definitions of distance, minor arc, major arc

Betweenness on the sphere

Given distinct points A, B with $B \neq A^*$, If X is a third point on $\overleftrightarrow{AB}$, then

$$\begin{array}{ll} X \text{ in } \overline{AB} \text{ [not an endpoint]} & \iff A - X - B \\ X \text{ in } \overline{BA^*} [x \neq B] & \iff A - B - X \\ X \text{ in } \overline{B^*A} [x \neq A] & \iff X - A - B \\ X \text{ in } \overline{A^*B^*} \text{ [not an endpoint]} & \iff \text{no betweenness} \end{array}$$

Furthermore, for any $X \neq A, A^*$, we have $A - X - A^*$

opposite hemispheres

angle measure using tangent vectors

No parallel line property: Any two lines in $\mathbb{S}$ meet in a pair of antipodal points.

In $\triangle ABC$ in $\mathbb{S}$, $m\angle A + m\angle B + m\angle C > 180$.

In particular, a triangle can have three right angles in $\mathbb{S}$.

Hyperbolic space $\mathbb{H}$:

distance definition using natural log [memorize this]

Betweenness Axioms: Given three distinct collinear points A, B, C , exactly one of them is between the other two.

angle measure in Appendix II, page 262 [you do not need to memorize this]

opposite halfplane

Given a line ℓ and a point P not on ℓ . Then there are infinitely many lines through P parallel to ℓ .

In $\triangle ABC$ in $\mathbb{H}$, $m\angle A + m\angle B + m\angle C < 180$.

PROOFS YOU SHOULD KNOW:

- (1) How to use “Two Points on a Ray” Theorem to prove uniqueness for the $\mathbb{R}\mathbb{R}$ axiom
- (2) Midpoint Theorem
- (3) How to use duality to prove theorems about pencils and angles
- (4) Angle Bisector Theorem
- (5) Why Taxicab Geometry fails SAS
- (6) Why ASS is not a triangle congruence criterion
- (7) How to use SAS to construct standard high school geometry proofs
- (8) Isosceles Triangle Theorem and its Converse
- (9) Vertical Angle Theorem
- (10) Why the Gap Space fails axioms S and C
- (11) Why 3-dimensional space $\mathbb{R}^3$ fails axiom S
- (12) The Perpendicular Line Theorem: Given a line ℓ and a point P , there is a line m through P that is perpendicular to ℓ
 Case (i) P on ℓ
 Case (ii) P is not on ℓ
- (13) The flaw in the proof that “all triangles are isosceles”
- (14) Why the Parallel Postulate cannot be proved from the other axioms of geometry
- (15) In Euclidean geometry, the sum of the measures of the angles of a triangle is 180, using the Alternate Interior Angle Theorem