

1. MATH 353

Theorem 8.3

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2. THEOREM 10.5

The Uniqueness of Antipodes Theorem

Hypothesis:

$$\omega < \infty$$

Line m separates $\mathbb{P}$ into two halfplanes H and K .

A is a point on line m

B satisfies $AB = \omega$.

By I3, A and B lie on some line n

Axiom N guarantees the existence of a point P such that

$$n = \overleftrightarrow{AP}, 0 < AP < \omega$$

$Q = P_n^*$, the antipode of P on line n

Conclusion:

B lies in line m .

If $m \neq n$, then the interiors of rays $\overrightarrow{AP}$ and $\overrightarrow{AQ}$ lie entirely in opposite halfplanes.

3. IMPORTANCE OF THEOREM 10.5

The hypothesis of Theorem 10.5 tells us

$$B = A_n^*$$

The conclusion of the Theorem $B \in m$ tells us

$$B = A_m^*$$

Thus, $A_n^* = A_m^*$.

In other words, the antipode of A is independent of the line m .

We denote the unique antipode of A (regardless of the line) by A^*

4. THE THREE STRATEGIES

Line m separates $\mathbb{P}$ into two halfplanes H and K .

Strategy 1. [“Two points on a line (Axiom I4) Argument”]

If line $n \neq m$ and A is a point on both lines m and n ,

and B is a point on n with $0 < AB < \omega$,

then B is not on line m .

Strategy 2. [Convexity Argument]

Given two point X and Y in halfplane H ,

the segment $\overline{XY}$ lies entirely in H .

Strategy 3. [Separation Argument]

If point X lies in halfplane H and point Y lies in K ,

then the segment $\overline{XY}$ crosses line m .

5. PROOF OF THEOREM 10.5

ω is finite

$$A \in m$$

$$AB = \omega$$

We wish to show $B \in m$.

There exists a line n containing A and B by I3.

Since $AB = \omega$, we know

$$B = A_n^* \text{ and } B \in n.$$

If $n = m$, we are done.

Assume $m \neq n$.

Axiom N guarantees a point $P \in n$ such that

$$0 < AP < \omega$$

6. PROOF CONTINUED

Let $Q = P_n^*$.

We know P cannot lie on m .

Why not?

Otherwise we would have two points A and P on two lines m and n with $0 < AP < \omega$, contradicting Axiom I4.

So P must lie on one of the halfplanes.

WLOG let's say P lies on halfplane H .

By Theorem 9.1,

$$\overrightarrow{AP} = \overline{AP} \cup \overline{PB}$$

$$\overrightarrow{AQ} = \overline{AQ} \cup \overline{QB}$$

7. FIVE CLAIMS

Claim 1: $A - X - P \implies X \in H$

Claim 2: $Q \in K$

Claim 3: $P - X - B \implies X \in H$

Claim 4: $A - X - Q \implies X \in K$

Claim 5: $Q - X - B \implies X \in K$

8. PROOF OF CLAIM 1

Claim 1: $A - X - P \implies X \in H$.

Pf of claim:

Either $X \in m$, $X \in H$, or $X \in K$.

We will rule out the first and the third possibilities.

$X \in m$

$\implies A, X \in m \cap n, AX < \omega$

contradiction

$X \in K$, together with $P \in H$

$\implies \overline{XP}$ meets m at some point Y in m

$\implies A, Y \in m \cap n, AY < \omega$

contradiction.

This proves the claim.

9. WHERE'S Q ?

Proof of Claim 2: $Q \in K$

Q on m

$\implies A$ and Q lie on $m \cap n$,

$\implies m = n$ (since $0 < AQ < \omega$), contradiction.

Suppose $Q \in H$

Pick any point satisfying $A - X - P$

By Claim 1, $X \in H$

By convexity, $\overline{QX} \subseteq H$

But $A \in \overline{QX}$ and $A \notin H$

contradiction.

10. MERCY

The proof of Claims 3 – 5 are similar.

They will not be given here, to keep you from going crazy.

You should verify them on your own.

11. OBSERVATIONS ABOUT INTERIORS

The five claims give us the following facts:

$\text{Int } \overrightarrow{AP} \subseteq H$

$\text{Int } \overrightarrow{AQ} \subseteq K$.

This is the second statement in the conclusion of Thm 10.5.

12. FIRST CONCLUSION OF THM 10.4

Pick a point X in the interior of $\overline{PB}$

and a point Y in the interior of $\overline{QB}$.

Then $P - X - B$ and $Q - Y - B$.

By Claims 3 and 5,

Then $X \in H$ and $Y \in K$.

Also $XY < \omega$.

By the separation axiom S, the segment $\overline{XY}$ meets line m .

But where?

13. GRAND FINALE

By the “Cut Segment” Theorem,

$$\overline{XY} = \overline{XB} \cup \overline{BY}$$

Moreover,

$$\overline{XB} - \{B\} \subseteq H \text{ and}$$

$$\overline{BY} - \{B\} \subseteq K.$$

The only point in $\overline{XY}$ which could lie on line m is B .

So $B \in m$

This completes the proof.

14. SECOND CONSEQUENCE OF THEOREM 10.5

Theorem. Suppose ω is finite

and m is a line in $\mathbb{L}$.

Let P be any point in $\mathbb{P}$ and $Q = P^*$

Then either P and Q both lie on line m or

P and Q lie on opposite halfplanes of m .

15. PROOF

To see this, let A be any point on m .

If P lies on m , then we are done since Theorem 10.5 tells us that Q also lies on m .

So assume P does not lie on m

and consider the line $\overleftrightarrow{AP}$.

WLOG say that P lies on halfplane H .

From the proof of Theorem 10.5 (Claim 2) we know

$Q \in K$.

16. WHY THIS IS A BIG DEAL?

Theorem. Suppose ω is finite.

Then any two distinct lines m and n must intersect.

Proof: By the Separation Axiom, m splits the plane into two halfplanes H and K .

Let P be any point on n .

If P lies on m , we are done.

If P does not lie on m , then P and $Q = P^*$ lie on opposite halfplanes, say $P \in H$ and $Q \in K$.

Let X be any point on n with $0 < PX < \omega$.

If X lies on line m , we are done.

Otherwise, X lies on halfplane H or K .

If X lies on K , then segment $\overline{PX}$ meets line m .

If X lies on H , then segment $\overline{QX}$ meets line m .

Either way, m and n intersect.

17. CAN YOU SPELL “PARALLEL”?

Definition:

Two lines m and n are **parallel** if and only if they do not intersect.

The conclusion of the last Theorem is

When ω is finite, there are no parallel lines.