

1. MATH 353

Theorem 6.4

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2. LENGTH OF A SEGMENT

Let S be the segment $\overline{AB}$.

We wish to define the length of segment S .

Can we just say that the length of S is AB , the length between the endpoints?

NO!

The problem is that, as unlikely as it seems, the endpoints may not be unique.

Example: In the Inside-Out Model, we have the surprising fact that points C and D lie inside segment $\overline{AB}$, while points A and B lie inside segment $\overline{CD}$.

So if $S = \overline{AB} = \{A, B, C, D\}$, the length of S is

(i) AB if we use endpoint A and B to write $S = \overline{AB}$

(ii) CD if we use endpoint C and D to write $S = \overline{CD}$.

How do we know these two lengths are the same?

3. THEOREM 6.4

Hypothesis:

$$\overline{AB} = \overline{CD}$$

Conclusion:

- (i) Either $\{A, B\} = \{C, D\}$ or $\{A, B\} \cap \{C, D\} = \emptyset$
- (ii) In either case, $AB = CD$

4. PRELIMINARIES

When we write the segment $\overline{AB}$ we are assuming

$$0 < AB < \omega$$

When we write the segment $\overline{CD}$ we are assuming

$$0 < CD < \omega$$

Since C and D lie on $\overline{AB}$ it follows that

A, B, C, D are collinear points (not necessarily distinct).

5. PROOF OF THM 6.4

There are three possibilities for how the sets $\{A, B\}$ and $\{C, D\}$ overlap:

either the intersection consists of two points, one point, or no points in common.

If $\{A, B\} \cap \{C, D\}$ contains two elements, then $\{A, B\} = \{C, D\}$, the first conclusion in (i).

If $\{A, B\} \cap \{C, D\}$ contains zero elements, then $\{A, B\} \cap \{C, D\} = \emptyset$, the second conclusion in (i).

So in order to prove statement (i) we need to show that the case where $\{A, B\} \cap \{C, D\}$ contains exactly one element doesn't happen.

6. RULING OUT THIS THIRD SUBCASE

We need to show that the case where $\{A, B\} \cap \{C, D\}$ contains exactly one element doesn't happen.

How do we show something doesn't happen?

By assuming it does happen and deriving a contradiction.

So assume that $\{A, B\} \cap \{C, D\}$ consists of exactly one point.

This can happen if four possible ways:

- (1) $A = C, B \neq D$
- (2) $A = D, B \neq C$
- (3) $B = C, A \neq D$
- (4) $B = D, A \neq C$.

7. CUTTING THE WORK BY 75%

Do we need to consider all four cases?

Happily the answer is: No!

The key is in the mathematical phrase

WLOG

which means

without loss of generality

You could switch $A \leftrightarrow B$ or $C \leftrightarrow D$ and the hypothesis of Theorem 6.4 would be the same.

So WLOG we may assume $A = C$ and $B \neq D$.

This saves 75% of the work, since we need not consider the other three cases.

We must, however, prove this case.

8. $A = C, B \neq D$

Consider point D .

$D \in \overline{AB}, D \neq A$, Why not? $D \neq B$, Why not?

So $A - D - B$

Now consider point B .

$B \in \overline{CD}, B \neq C$, Why not? $B \neq D$, Why not?

So $C - B - D$

which implies $A - B - D$ since $C = A$

But $A - D - B$ and $A - B - D$ violates Unique Middle.

9. CONCLUSION (II) OF THM 6.4

It remains to prove that (ii) $AB = CD$.

If $\{A, B\} = \{C, D\}$, then the conclusion follows easily:

If $A = C$ and $B = D$, then $AB = CD$ holds by substitution.

If $A = D$ and $B = C$, then $AB = CD$ holds by the symmetry axiom of distance.

Now assume that $\{A, B\} \cap \{C, D\} = \emptyset$.

10. TWO EQUATIONS

C, D lie inside $\overline{AB}$

(meaning neither is endpoint A or B)

$A - C - B$ and $A - D - B$.

By definition of betweenness,

$AC + CB = AB$ and $AD + DB = AB$.

Adding these two equations gives us

$$(*) \ AC + CB + AD + DB = 2AB.$$

A, B lie inside $\overline{CD}$

(meaning neither is endpoint C or D)

$$C - A - D \text{ and } C - B - D.$$

By definition of betweenness,

$$CA + AD = CD \text{ and } CB + BD = CD.$$

Adding these two equations gives us

$$(**) \ CA + AD + CB + BD = 2CD.$$

11. FINAL PIECE OF THE PROOF

So far we have two equations

$$(*) \ AC + CB + AD + DB = 2AB.$$

and

$$(**) \ CA + AD + CB + BD = 2CD.$$

By the symmetry of distance axiom and algebra,
the left hand sides of $(*)$ and $(**)$ are the same.

Hence

$$2AB = 2CD.$$

Dividing by 2 gives equation (ii) in Theorem 6.4:

$$AB = CD$$

This completes the proof.

12. SO WHAT?

It now makes sense to define the length of $\overline{AB}$ as AB .

For even if we could write this segment using two other endpoints $\overline{CD}$, Theorem 6.4 tells us that we will get the same value using C and D as using A and B .