

1. MATH 353

Theorem 7.5

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2. THEOREM 7.5

Hypothesis:

X, Y, A are three distinct points.

X, Y lie on ray $\overrightarrow{AB}$

Conclusion:

(i) $A - X - Y$

(ii) $A - Y - X$

or (iii) X, Y lie inside segment $\overline{AB}$, that is, $A - X - B$ and $A - Y - B$.

3. HANDLE THE CASE $X = B$

Suppose $X = B$

Since $Y \in \overrightarrow{AB}$

Four possibilities:

$Y = A$

impossible since $\{A, X, Y\}$ are distinct

$A - Y - B \implies A - Y - X$

(since $B = X$)

$Y = B \implies Y = X$ (since $X = B$)

This violates the hypothesis that $\{A, X, Y\}$ are distinct

$A - B - Y \implies A - X - Y$

(since $B = X$)

4. HANDLE THE CASE $Y = B$

A similar argument to the one above leads to $A - X - Y$ or $A - Y - X$

At this point we have that $\{A, B, X, Y\}$ are four distinct points and X, Y lie on $\overrightarrow{AB}$.

5. THE FOUR SUBCASES

Since $X \neq A, B$ and $X \in \overrightarrow{AB}$

either $A - X - B$ or $A - B - X$.

Since $Y \neq A, B$ and $Y \in \overrightarrow{AB}$

either $A - Y - B$ or $A - B - Y$.

This gives us four subcases:

(i) $A - X - B$ and $A - Y - B$

(ii) $A - X - B$ and $A - B - Y$

(iii) $A - B - X$ and $A - Y - B$

(iv) $A - B - X$ and $A - B - Y$

6. CASE (I)

(i) $A - X - B$ and $A - Y - B$

In this case both X and Y lie in $\overline{AB}$.

This is the third option of our hypothesis.

Done.

7. CASE (II)

(i) $A - X - B$ and $A - B - Y$

By the insertion axiom

$$A - X - B - Y$$

This four-way betweenness implies $A - X - Y$.

Done.

8. CASE (III)

(i) $A - B - X$ and $A - Y - B$

By the insertion axiom

$$A - Y - B - X$$

This four-way betweenness implies $A - Y - X$.

Done.

9. CASE (IV)

(i) $A - B - X$ and $A - B - Y$

By Quadrachotomy, using Y as the fourth point in the betweenness $A - B - X$,

(a) $Y - A - B$,

(b) $A - Y - B$,

(c) $B - Y - X$,

(d) $B - X - Y$,

Both (a) and (b) violate Unique Middle Theorem since we know $A - B - Y$.

This leaves (c) and (d).

10. SUBCASE (c) $B - Y - X$

Use Insertion with $A - B - X$ to get

$$A - B - Y - X$$

This four-way betweenness implies $A - Y - X$

Done

11. SUBCASE (d) $B - X - Y$

Similar to subcase (c).

Use Insertion with $A - B - Y$ to get

$$A - B - X - Y$$

Thus $A - X - Y$

This completes the proof.