

1. MATH 353

Theorem 8.3

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2. THEOREM 8.3

The “Three Points on a Ray” Theorem.

Hypothesis:

X, Y, A are three distinct points

X, Y lie on ray $\overrightarrow{AB}$

Conclusion:

(i) $A - X - Y$ or (ii) $A - Y - X$

3. CONSEQUENCE 1

Theorem [“Uniqueness of the Point in $\mathbb{RR}$ ”]

The point X on ray $\overrightarrow{AB}$ in the Real Ray Axiom whose distance from A is $AX = s$, $0 \leq s \leq \omega$, is **unique**.

4. PROOF

Suppose there is a second point Y on $\overrightarrow{AB}$ such that $AX = AY = s$.

It is easy to handle the case where $s = 0$, since $AX = AY = 0$ implies $A = X = Y$.

So assume $s \neq 0$.

Then A , X , and Y are three distinct points on $\overrightarrow{AB}$.

By Theorem 8.3, $A - X - Y$ or $A - Y - X$.

In the first case,

$$A - X - Y \implies AX + XY = AY$$

$$\implies XY = 0 \text{ (since } AX = AY)$$

$$\implies X = Y, \text{ a contradiction.}$$

In the second case,

$$A - Y - X \implies AY + YX = AX \implies YX = 0 \implies X = Y, \text{ a contradiction.}$$

5. CONSEQUENCE 2

The “Equivalent Ray” Theorem.

Hypothesis:

C lies on $\overrightarrow{AB}$, $0 < AC < \omega$

Conclusion:

$$\overrightarrow{AB} = \overrightarrow{AC}$$

that is, we may use any second point C on the ray $\overrightarrow{AB}$

other than $C = A$ and $C = A^*$ (when ω is finite)

to describe the ray.

6. PROOF OF CONSEQUENCE 2

We begin with an observation:

$C \neq A$ since $AC > 0$.

To show the two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are equal

we must show that each is a subset of the other:

$$\overrightarrow{AB} \subseteq \overrightarrow{AC}$$

and

$$\overrightarrow{AC} \subseteq \overrightarrow{AB}.$$

7. SHOW $\overrightarrow{AB} \subseteq \overrightarrow{AC}$

Pick a point $X \in \overrightarrow{AB}$

We want to show $X \in \overrightarrow{AC}$

If $X = A$, then

$$X \in \overrightarrow{AC}$$

since $X = A$ is the endpoint A of ray $\overrightarrow{AC}$.

Done

If $X = C$, then

$$X \in \overrightarrow{AC}$$

since $X = C$ is an endpoint of $\overline{AC} \subseteq \overrightarrow{AC}$.

Done

8. SHOW $X \in \overrightarrow{AC}$

This leaves the situation where A, C, X are distinct points on $\overrightarrow{AB}$.

By Theorem 8.3, either

$A - X - C$ or

$A - C - X$.

In either case, we have

$$X \in \overrightarrow{AC}.$$

Done

9. SHOW $\overrightarrow{AC} \subseteq \overrightarrow{AB}$

Again we know $A \neq C$.

We would like to switch letters B and C and use the proof we just gave.

In that proof we assumed $C \in \overrightarrow{AB}$ and showed

$$X \in \overrightarrow{AB} \implies X \in \overrightarrow{AC}.$$

We can reverse the argument **provided** we can show that

$$B \in \overrightarrow{AC}.$$

10. SHOW $B \in \overrightarrow{AC}$

From $C \in \overrightarrow{AB}$, $0 < AC < \omega$ we know

(i) $A - C - B$

(ii) $C = B$ or

(iii) $A - B - C$

In all three of these cases, it follows that

$$B \in \overrightarrow{AC}.$$

This completes the proof.

11. CONSEQUENCE 3

The “Segment Cut” Theorem.

Suppose $A - B - C$.

Then $\overline{AB} \cup \overline{BC} = \overline{AC}$.

Once again, to show two sets are equal, we must show that they are subsets of each other:

$$\overline{AB} \cup \overline{BC} \subseteq \overline{AC} \text{ and}$$

$$\overline{AC} \subseteq \overline{AB} \cup \overline{BC}.$$

$$12. \overline{AB} \cup \overline{BC} \subseteq \overline{AC}$$

Pick any X in $\overline{AB}$.

If X is either A or B then X lies in $\overline{AC}$ since both A and B lie in this segment.

So assume $A - X - B$.

Then by insertion with $A - B - C$, we get

$$A - X - B - C$$

Dropping the B gives us

$$A - X - C$$

So $X \in \overline{AC}$.

This proves $\overline{AB} \subseteq \overline{AC}$.

By a similar argument, we can show

$$\overline{BC} \subseteq \overline{AC}.$$

Thus,

$$\overline{AB} \cup \overline{BC} \subseteq \overline{AC}.$$

$$13. \overline{AC} \subseteq \overline{AB} \cup \overline{BC}$$

Pick any X in $\overline{AC}$.

If X is A , B , or C then clearly X lies in $\overline{AB} \cup \overline{BC}$ since both A , B , and C lie in $\overline{AB} \cup \overline{BC}$.

So assume X is distinct from A , B , and C .

Then $A - X - C$ since $X \in \overline{AC}$ and $X \neq A$ or C .

X and B are distinct points other than A on $\overline{AB}$

But $\overline{AB} \subseteq \overrightarrow{AB}$.

So X and B are distinct points other than A on $\overrightarrow{AB}$.

14. $\overline{AC} \subseteq \overline{AB} \cup \overline{BC}$ CONTINUED

By Theorem 8.3, either

$A - X - B$ or $A - B - X$.

If $A - X - B$ then

$X \in \overline{AB}$.

If $A - B - X$ then by insertion with $A - X - C$ we have

$A - B - X - C$

$\implies B - X - C$.

$\implies X \in \overline{BC}$.

Either way,

$X \in \overline{AB} \cup \overline{BC}$.

Thus, $\overline{AC} \subseteq \overline{AB} \cup \overline{BC}$.

This completes the proof.

15. CONSEQUENCE 4

Theorem [“Opposite Ray Theorem”]

Assume $A - B - C$. Then $\overrightarrow{BA} \cup \overrightarrow{BC} = \overleftarrow{AB}$.

Rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are called **opposite** rays.

In general, two rays on a line ℓ are called opposite rays if they share an endpoint and their union is line ℓ .

16. PROOF OF OPPOSITE RAY THEOREM

Assume $A - B - C$.

To show two sets are equal

$$\overrightarrow{BA} \cup \overrightarrow{BC} = \overleftrightarrow{AB},$$

we must show they are subsets of each other:

$$(1) \overrightarrow{BA} \cup \overrightarrow{BC} \subseteq \overleftrightarrow{AB},$$

and

$$(2) \overleftrightarrow{AB} \subseteq \overrightarrow{BA} \cup \overrightarrow{BC}.$$

Relation (1) follows immediately from the fact that

both rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ lie on the line $\overleftrightarrow{AB}$.

$$17. \overleftrightarrow{AB} \subseteq \overrightarrow{BA} \cup \overrightarrow{BC}$$

Pick any point X on $\overleftrightarrow{AB}$.

If X is A , B , or C then $X \in \overrightarrow{BA} \cup \overrightarrow{BC}$.

So assume that X , A , B , and C are distinct.

Since they are also collinear and $A - B - C$, the Quadrichotomy Axiom tells us that one of four things happen:

$$(a) X - A - B$$

$$(b) A - X - B$$

$$(c) B - X - C$$

$$(d) B - C - X$$

Choices (a) and (b) imply $X \in \overrightarrow{BA}$.

Choices (c) and (d) imply $X \in \overrightarrow{BC}$.

Either way, $X \in \overrightarrow{BA} \cup \overrightarrow{BC}$.

18. REMARK

Note we did not use Thm 8.3 in this proof.

All we needed was the Quadrichotomy Axiom.

19. CONSEQUENCE 5

Theorem [“Unique Endpoints”]

Assume C and D are distinct points in $\overline{AB}$. If $\{A, B\} \neq \{C, D\}$, then $CD < AB$.

Moreover, $\overline{AB} = \overline{CD}$ if and only if $\{A, B\} = \{C, D\}$.

This means that the endpoints of a segment are unique, except for order.

The proof is based on Thm 6.4 and the following lemma.

20. LEMMA

If $X \in \overline{AB}$, then $AX \leq AB$.

Proof:

If $X = A$, then $AX = 0 < AB$.

If $X = B$, then $AX = AB$.

If $A - X - B$, then

$$AX + XB = AB$$

$$\implies AX < AB.$$

This completes the proof of the lemma.

21. PROOF OF CONSEQUENCE 5

We consider two cases:

Case 1. $C = A, D \neq B$

(when the sets $\{A, B\}$ and $\{C, D\}$ overlap in one point).

The other cases $C = B, D \neq A$, etc., are similar.

Case 2. A, C, D are distinct.

22. CASE 1: $C = A, D \neq B$

Since $D \neq B$, we know

$A = C, B, D$ are distinct collinear points and

$$D \in \overline{AB}$$

Thus,

$$A - D - B$$

$$\implies AD + DB = AB$$

$$\implies AD < AB$$

$$\implies CD < AB \text{ [since } C = A]$$

23. CASE 2: A, C , AND D ARE DISTINCT

Since both C and D lie on $\overrightarrow{AB}$, either

$$A - C - D \text{ or } A - D - C.$$

$$A - C - D$$

$$\implies AC + CD = AD$$

$$\implies CD < AD \leq AB \text{ by the lemma}$$

$$A - D - C$$

$$\implies AD + DC = AC$$

$$\implies CD = DC < AC \leq AB \text{ by the lemma.}$$

This proves the first statement

$$CD < AB.$$

24. THEOREM 6.4

Hypothesis:

$$\overline{AB} = \overline{CD}$$

Conclusion:

(i) Either $\{A, B\} = \{C, D\}$ or $\{A, B\} \cap \{C, D\} = \emptyset$

(ii) In either case, $AB = CD$

25. USE OF THEOREM 6.4

Assume $\overline{AB} = \overline{CD}$

According to Theorem 6.4,

(i) Either $\{A, B\} = \{C, D\}$ or $\{A, B\} \cap \{C, D\} = \emptyset$

and

(ii) $AB = CD$

If the second part of (i) holds, that is, if

$$\{A, B\} \cap \{C, D\} = \emptyset$$

then by the first part of our new theorem, $CD < AB$, contradicting statement (ii).

Thus, the first part of (i) must hold:

$$\{A, B\} = \{C, D\}$$

This completes the proof.

26. INTERIOR OF A SEGMENT

The **interior** of segment $\overline{AB}$,

denoted $\overline{AB}^\circ$

is the set $\overline{AB}$ minus its endpoints, that is,

$$\overline{AB}^\circ = \overline{AB} - \{A, B\}$$