

1. MATH 353

Theorem 8.3

Professor Richard Blecksmith

richard@math.niu.edu

Dept. of Mathematical Sciences

Northern Illinois University

2. THEOREM 8.3

Hypothesis:

X, Y, A are three distinct points

X, Y lie on ray $\overrightarrow{AB}$

Conclusion:

(i) $A - X - Y$ or (ii) $A - Y - X$

3. THEOREM 7.5

Hypothesis:

X, Y, A are three distinct points

X, Y lie on ray $\overrightarrow{AB}$

Conclusion:

(i) $A - X - Y$

(ii) $A - Y - X$

(iii) X, Y lie inside $\overline{AB}$

4. MAKE USE OF PROP 7.5

Assume that the hypothesis of Theorem 8.3 is true.

By Prop 7.5, conditions (i), (ii), or (iii) must hold.

Conditions (i) and (ii) are the conclusion we want in Thm 8.3.

So it only remains to examine the case where condition (iii) holds:

X, Y lie inside $\overline{AB}$

$A - X - B$ and $A - Y - B$

5. APPLY THE REAL RAY AXIOM

Pick $s = AB + \frac{\omega - AB}{2} = \frac{AB + \omega}{2}$ if ω is finite.

Pick $s = AB + 1$ if $\omega = \infty$.

By the Axiom RR, there exists a point E on $\overrightarrow{AB}$ such that $AE = s$.

There are four possibilities:

(i) $E = A$

(ii) $A - E - B$

(iii) $E = B$

(iv) $A - B - E$

6. RULE OUT THE FIRST THREE SUBCASES

(i) $E = A$ is impossible since $EA = s > 0$

(ii) $A - E - B$ implies $AE < AB$ which is impossible since $AE = s$ and $s > AB$

(iii) $E = B$ implies $AE = AB$ which is impossible since $AE = s$ and $s > AB$

This leaves the last choice $\boxed{A - B - E}$

7. USE INSERTION

$A - X - B$ and $A - B - E$

$\implies A - X - B - E$

$\implies X - B - E$

$\implies E - B - X$

Similarly

$A - Y - B$ and $A - B - E$

$\implies A - Y - B - E$

$\implies Y - B - E$

$\implies E - B - Y$

8. CONSIDER RAY FROM E THROUGH B

From $E - B - X$ and $E - B - Y$ we know

X, Y lie on $\overrightarrow{EB}$

We also know E, X , and Y are distinct.

Why?

Apply Prop 7.5 to points E, X, Y and ray $\overrightarrow{EB}$ to get

(i) $E - X - Y$

(ii) $E - Y - X$

(iii) X, Y lie in $\overline{EB}$

9. ELIMINATE THE CASE (III)

(iii) X, Y lie inside $\overline{EB}$

That is, X and Y are not endpoints and lie on segment $\overline{EB}$

By definition of segment,

$$E - X - B$$

But we have already shows that $E - B - X$

By the Law of Unique Middle $E - X - B$ is impossible.

Hence case (iii) is impossible.

10. THE GRAND FINALE

What we know so far:

- $A - X - E$ from $A - X - B - E$
- $A - Y - E$ from $A - Y - B - E$
- $E - X - Y$ or $E - Y - X$

Suppose $E - X - Y$ is true.

Use insertion with $A - Y - E$ to get

$$A - Y - X - E$$

$$\implies A - Y - X.$$

Suppose $E - Y - X$ is true.

Use insertion with $A - X - E$ to get

$$A - X - Y - E$$

$$\implies A - X - Y.$$

This completes the proof.