

1. MATH 353

Theorem 8.3

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2. THEOREM 9.1

The Antipode Theorem

Hypothesis:

$$\omega < \infty$$

A is a point on line ℓ

Conclusion:

There exists a unique point A_ℓ^* such that $AA_\ell^* = \omega$.

For any point P on line ℓ , $P \neq A, A_\ell^*$, we have a betweenness relation

$$A - P - A_\ell^*.$$

3. START OF PROOF

The Nontriviality Axiom guarantees that a point B lies on ℓ with $0 < AB < \omega$.

The Real Ray Axiom guarantees that there exists a point C on ray $\overrightarrow{BA}$ such that $BC > BA$.

Pick $s = AB + \frac{1}{2}(\omega - AB)$.

It is easy to see that $B - A - C$.

By the Opposite Ray Theorem,

$$\ell = \overrightarrow{AB} \cup \overrightarrow{AC}$$

4. EXISTENCE OF ANTIPODES

The Real Ray Axiom guarantees that there exists a point X on ray $\overrightarrow{AB}$ such that $AB = \omega$.

By the Uniqueness Theorem, X is the only point on ray $\overrightarrow{AB}$ with this property.

But how do we know that there isn't another different point Y on line ℓ whose distance from A is $AY = \omega$?

By the Opposite Ray Theorem, this second point Y must lie on the ray opposite to $\overrightarrow{AB}$,

that is $Y \in \overrightarrow{AC}$.

5. DISTANCE XY

Since X is on ray $\overrightarrow{AB}$, we know

(i) $X = A$

(ii) $A - X - B$

(iii) $X = B$

or (iv) $A - B - X$

Cases (i), (ii), and (iii) lead to immediate contradictions.

Therefore, $A - B - X$.

6. PROOF CONTINUED

We have $A - B - X$.

Assuming that Y is a fourth point on ℓ , then Quadrichotomy says that Y must fit in somewhere

$$(1) \ Y - A - B$$

$$(2) \ A - Y - B$$

$$(3) \ B - Y - X$$

$$(4) \ B - X - Y$$

$$(1) \implies YA + AB = YB$$

$$\implies \omega + AB = YB$$

$$\implies YB > \omega, \text{ a contradiction}$$

$$(2) \implies AY + YB = AB \implies AB > AY = \omega,$$

a similar contradiction.

7. EXISTENCE OF $\overline{XY}$

This leaves cases

$$B - Y - X \text{ and } B - X - Y$$

$$B - Y - X$$

$$\implies BY + YX = BX$$

$$\implies YX < \omega$$

Similarly,

$$B - X - Y \implies XY < \omega$$

Since we are assuming $X \neq Y$,

we know $XY > 0$

Thus, $0 < XY < \omega$

and $\overline{XY}$ is a legitimate segment.

8. THE MIDPOINT ARGUMENT

Let M be the midpoint of $\overline{XY}$.

$$X - M - Y$$

The problem is: where is A ?

Check that $A \neq X, Y, M$

By the Quadrichotomy Axiom,

$$(1) A - X - M$$

$$(2) X - A - M$$

$$(3) M - A - Y$$

$$(4) M - Y - A$$

All four of these betweenness relations lead to contradictions.

For example,

$$A - X - M \implies AX + XM = AM$$

$$\implies AM > \omega, \text{ a contradiction.}$$

The other cases are similar.

9. PROOF THAT $X = Y$

This contradiction arose from the assumption $X \neq Y$.

It must, therefore, be true that $X = Y$.

That is, the point X on $\overrightarrow{AB}$ whose distance from A is ω equals the point Y on the opposite ray $\overrightarrow{AC}$ whose distance from A is ω .

10. TERMINOLOGY BREAK

When $\omega < \infty$, the unique point X on ℓ such that $AX = \omega$ is called the **antipode** of A on ℓ .

It is written using a superscript star:

$$X = A_\ell^*$$

For now, it is dependent on the line ℓ .

If we change to another line m through A , then possibly the antipode for that different line A_m^* could be different than A_ℓ^* .

Later in the course we shall prove that this doesn't happen.

The antipode of A is the same no matter what line we use.

11. GRAND FINALE OF PROOF

By the Opposite Ray Theorem, we know

$$\overleftrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{AC}$$

For any point P on line ℓ other than A , X , or Y , we know

If P lies on $\overrightarrow{AB}$, then $A - P - X$

while if P lies on $\overrightarrow{AC}$, then $A - P - Y$

But $X = Y = A_\ell^*$, so in either case we have

$$A - P - A_\ell^*.$$

12. MORE OBSERVATIONS ABOUT RAYS WHEN $\omega < \infty$

Assume ω is finite.

Suppose $B - A - C$.

From the Opposite Ray Theorem, we know

$$\overleftrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{AC}$$

$$\overrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{BA}_\ell^*$$

$$\overrightarrow{AC} = \overrightarrow{AC} \cup \overrightarrow{CA}_\ell^*$$

13. ENDPOINTS AND INTERIOR OF A RAY

When ω is finite,

$$\overrightarrow{AB} = \overrightarrow{A_\ell^* B}$$

Thus,

$\overrightarrow{AB}$ has two endpoints:

A and A_ℓ^*

When $\omega = \infty$,

the ray $\overrightarrow{AB}$ has only one endpoint: the point A .

The **interior** of ray $\overrightarrow{AB}$,

denoted $\overrightarrow{AB}^\circ$

is the set $\overrightarrow{AB}$ minus its endpoints, that is,

$$\overrightarrow{AB}^\circ = \begin{cases} \overrightarrow{AB} - \{A, A_\ell^*\} & \text{if } \omega < \infty \\ \overrightarrow{AB} - \{A\} & \text{if } \omega = \infty \end{cases}$$

14. ALMOST UNIQUENESS FOR QUADRICHOTOMY

Given $A - B - C$, $\omega < \infty$, and B^* = the antipode of B on line $\ell = \overleftrightarrow{AB}$.

$$\overleftrightarrow{AB} = \overline{B^*A} \cup \overline{AB} \cup \overline{BC} \cup \overline{CB^*}$$

If X lies on line ℓ and $X \neq A, B, C, B^*$, then **exactly** one of the following holds:

$$(1) \ X - A - B \quad \text{when } X \in \overline{B^*A}$$

$$(2) \ A - X - B \quad \text{when } X \in \overline{AB}$$

$$(3) \ B - X - C \quad \text{when } X \in \overline{BC}$$

$$(4) \ B - C - X \quad \text{when } X \in \overline{CB^*}$$

Note B^* satisfies (1) and (4).

15. FUSSY DETAILS

Given $A - B - C$, $\omega < \infty$, and B^* = the antipode of B on line $\ell = \overleftrightarrow{AB}$.

$$\overleftrightarrow{AB} = \overline{B^*A} \cup \overline{AB} \cup \overline{BC} \cup \overline{CB^*}$$

To see that $X \in \overline{B^*A} \implies X - A - B$

Assume $X \in \overline{B^*A}$.

We are told that $X \neq A, B, C, B^*$

So X is not an endpoint of $\overline{B^*A}$

It follows that X must be an interior point of $\overline{B^*A}$.

That is, $B^* - X - A$

By Thm 9.1, we know $B^* - A - B$

By insertion, $B^* - X - A - B$

and thus $X - A - B$.

The case $X \in \overline{CB^*} \implies B - C - X$ is similar.