

1. Beads on a Necklace

John's 4th grade teacher told him that points on a line are like beads on a necklace.
 Since beads touch each other, John assumed that points could touch each other.
 What do you think?
 Can two points touch each other?

2. The 999 Question

Does $.999\ldots = 1$?
 If you answered "no," then do these points touch each other?
 Tell me a point between them?

3. Resolving the 999 Question

Does $.999\ldots = 1$?
 If you are in a class, take a vote.
 If you are reading this by yourself, take a stance.
 If it were possible for points to touch, then $.999\ldots$ and 1 would be a likely example.
 The problem is: Distinct points do not touch.
 There are points in between them.
 The midpoint

$$M = \frac{1}{2}(P + Q),$$

for example, lies between P and Q .
 Indeed, if $.999\ldots \neq 1$, tell me any point between them!

4. Baseball analogy

It's like a ball thrown to first base that arrives in the first baseman's glove at exactly the same time as the batter's shoe touches the base.

5. Another Argument

Start with the equation $\frac{1}{3} = .3333\ldots$.
 Multiply by 3:
 $3 \times \frac{1}{3} = 1$
 $3 \times .3333\ldots = .9999\ldots$

6. A Third Argument

Let $x = .9999\ldots$.
 Then $10x = 9.999\ldots$ since multiplying by 10 just moves the decimal point one digit to the right.
 Subtract:
 $10x - x = 9.9999\ldots - .9999\ldots$
 $9x = 9$
 $x = 1$

7. Zeno

A version of Question 999 first got air time over two thousand years ago thanks to a mathematical prankster named Zeno, who went around confusing people with his riddles.

Zeno's Paradox, roughly translated, is

"In order to travel from your couch to the door, you must first get to the halfway point, from there you must travel to the second halfway point, then to the third halfway point, and so on. Since there are infinitely many halfway points, you can never reach the door."

Zeno, being male, expounded the ultimate "guy" belief: if you can't get off the couch, then you might as well lay back and watch some football. (Of course, somebody needs to bring the snacks.)

9. Binary real numbers

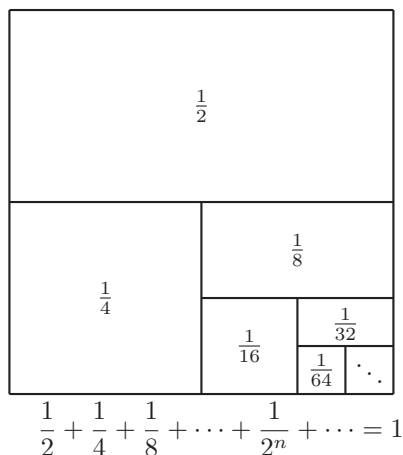
Perhaps the best way to understand Zeno's Paradox is to use binary (or base 2) notation. If each of the binary digits (or *bits*) $b_1, b_2, b_3, \dots$ is either 0 or 1, then the binary number $0.b_1b_2b_3\dots$ base 2 is defined to be

$$b_1 \cdot \frac{1}{2} + b_2 \cdot \frac{1}{4} + b_3 \cdot \frac{1}{8} + \dots + b_n \cdot \frac{1}{2^n} + \dots$$

The positions after the binary point are powers of $\frac{1}{2}$, just as in decimal notation, positions after the decimal point are powers of $\frac{1}{10}$. Thus $L = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ has the binary representation

$$L = 0.1111\dots \text{ (base 2).}$$

11. Proof Without Words



8. Infinite sums

Zeno was correct in asserting that the distance from the couch to the door is an infinite sum:

$$L = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots$$

But behind Zeno's Paradox is the assumption that the sum of infinitely many positive real numbers must add to infinity.

Right?

No, it's not right.

Decimal expansions are infinite sums:

$$\pi = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \frac{9}{100000} + \dots$$

and no one these days thinks that the infinite sum on the right is infinity.

10. The 111 Question

Zeno's Paradox boils down to

Question 111: Does $.111\dots = 1$ (base 2)?

A little thought shows us that Question 111 is the binary equivalent of Question 999 for decimals.

Consider the infinite sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots$$

What happens when we add up all the terms out to infinity? The argument is exactly the same as the argument in the 999 problem. The sum never exceeds one. On the other hand, we get in trouble if we say the sum is a number r less than 1, since eventually—using enough terms—the sum will exceed r . The only conclusion left is that the sum is exactly one, Zeno was wrong, and people can get off the couch, at least if they want to.