

Exploring Fractions

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Beads on a Necklace

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What do you think? Can two points touch each other?

The 999 Question

Does $.999 \dots = 1$?

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If you answered “no,” then do these points touch each other?

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If you answered “no,” then do these points touch each other?

Tell me a point between them?

Victoria's Problem

Victoria was asked to find a fraction between $\frac{2}{5}$ and $\frac{3}{7}$.

Fifth Grade Textbook Solution

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Add the numerator and denominator to get

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Is this answer correct?

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So

$$\frac{2}{5} < \frac{5}{12} < \frac{3}{7}$$

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How many fractions lie between $\frac{2}{5}$ and $\frac{3}{7}$?

General Method

It turns out that this method always works and that,

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It turns out that this method always works and that, in some sense, it is the very best possible answer.

Victoria's Theorem

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If $\frac{a}{b} < \frac{c}{d}$ are two positive fractions, then

$$\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}.$$

Experiment

Split up into groups of 3 or 4.

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Each group should think of their own example of two positive fractions.

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Test whether Victoria's Theorem is true for your example.

Challenge Proof

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$$\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}.$$

You may try more examples, where you replace the letters with actual numbers.

But the **proof** requires you to use just the letters.

The Mediant

Definition. If $\frac{a}{b}$ and $\frac{c}{d}$ are positive fractions, then

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The Median

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$\frac{a+c}{b+d}$ is called their **median**.

We will write

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.$$

Danger – Danger

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The point is that $\frac{a}{b} \oplus \frac{c}{d}$ is **not the sum** of the fractions; it is the mediant.

We must be careful never to confuse $\oplus$ with $+$.

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This means the batter got 3 hits out of 5 official at-bats in Game 1 and 1 hit out of 3 official at-bats in Game 2.

What is the batter's overage over both games?

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Context

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In this example we add the fractions as usual.

The Brady Bunch

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We add: $3 + 3 = 6$

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“Of course not,” replied the little boy. “There are 9 tiles one way, and 11 the other.”

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We multiply: $9 \times 11 = 99$

Mental Challenge

Can you think of a mathematical problem that requires

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For example, $\frac{10}{20} = \frac{1}{2}$.

In our baseball example, are $\frac{10}{20}$ and $\frac{1}{2}$ the same?

Does $\frac{10}{20} \oplus \frac{1}{3} = \frac{1}{2} \oplus \frac{1}{3}$?

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$$S = \left\{ \frac{109}{83}, \frac{2}{91}, \frac{108}{1237}, \frac{23107}{30} \right\}$$

Closest fractions

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The numerator $bc - ad$ is positive Why? and is an integer.

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So the smallest value for the $bc - ad$ is 1.

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Summary

Given two fractions $\frac{a}{b} < \frac{c}{d}$, the closest distance that they can be from each other is

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This minimum distance occurs when

$$bc - ad = 1.$$

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Two fractions $\frac{a}{b} < \frac{c}{d}$ which satisfy $ad + 1 = bc$ are called **contiguous**.

Note that $\frac{a}{b} + \frac{1}{bd} = \frac{c}{d}$

Tape Measures

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$$\text{So } \frac{1}{3} = \frac{5.3333 \dots}{16}$$

It looks like the closest fraction to $\frac{1}{3}$ is $\frac{5}{16}$, a mark on a tape measure.

Check it out!

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$$\frac{5}{16} < \frac{1}{3} < \frac{3}{8}$$

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Are these fractions contiguous?

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Look at the equation

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Are these fractions contiguous?

How do you suppose a carpenter would mark off one third of an inch?

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Discussion Problem

How are the fractions $\frac{a}{b} = \frac{2}{5}$ and $\frac{c}{d} = \frac{3}{7}$ related to the Pails of Water Problem for 5 and 7?

Exploration

If $\frac{a}{b} = \frac{2}{5}$ and $\frac{c}{d} = \frac{3}{7}$, show that the mediant $\frac{a}{b} \oplus \frac{c}{d}$ is contiguous to both $\frac{a}{b}$ and $\frac{c}{d}$.

Exploration

If $\frac{a}{b} = \frac{2}{5}$ and $\frac{c}{d} = \frac{3}{7}$, show that the mediant $\frac{a}{b} \oplus \frac{c}{d}$ is contiguous to both $\frac{a}{b}$ and $\frac{c}{d}$.

Split up into groups of 3 or 4.

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Each group should think of their own example of two contiguous fractions $\frac{a}{b}$ and $\frac{c}{d}$.

Is it true that the mediant is always contiguous to the endpoints $\frac{a}{b}$ and $\frac{c}{d}$ for the example you thought up?

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I challenge each group to prove this theorem.

There's more

It turns out that if $\frac{a}{b} < \frac{c}{d}$ are contiguous, then the
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For example, take the fractions $\frac{2}{5}$ and $\frac{3}{7}$.

They are contiguous and their mediant is $\frac{2+3}{5+7} = \frac{5}{12}$.

So 12 is the smallest denominator possible for any fraction between $\frac{2}{5}$ and $\frac{3}{7}$.

A direct check

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$$\frac{1}{3} < \frac{2}{5} \text{ and } \frac{2}{3} < \frac{3}{7}.$$

So neither $\frac{1}{3}$ nor $\frac{2}{3}$ is between them.

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$d = 10$: Show that no fraction with denominator 10 lies between $\frac{2}{5}$ and $\frac{3}{7}$.

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You might try some examples, like $\frac{a}{b} = \frac{1}{3}$ and $\frac{c}{d} = \frac{2}{3}$.

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So we start with $\mathcal{F}_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}$.

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So $\mathcal{F}_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$.

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Theorem. $\mathcal{F}_n$ consists of all reduced fractions $\frac{a}{b}$ in the interval $[0, 1]$, whose denominator b is less than or equal to n , arranged in increasing order.

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For homework, find

$$\mathcal{F}_9$$

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Hint 1: Pails of Water will get one solution.

Hint 2: Use the mediant equation to get the other solution.